

Year 2023-2024

The Chinese University of Hong Kong
Department of Mathematics
MMAT 5340 Probability and Stochastic Analysis

Homework 4: Stopping Times

Due Date: 23:59 pm on Tuesday, February 20th, 2024.
Please submit your homework on Blackboard

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_n)_{n \geq 0}$.

(a) Let τ_1, τ_2 be two \mathbb{F} -stopping times. Prove that

$$\begin{aligned}\tau_1 \wedge \tau_2 &:= \min(\tau_1, \tau_2) \\ \tau_1 \vee \tau_2 &:= \max(\tau_1, \tau_2)\end{aligned}$$

are both stopping times.

(b) Let τ be an \mathbb{F} -stopping time. Prove that $\tau + 1$ is also an \mathbb{F} -stopping time.

2. Let $X_0 = 0, X_n = \sum_{k=1}^n \xi_k$, where $(\xi_k)_{k \geq 1}$ is a sequence of independent and identically distributed random variables such that $\mathbb{P}[\xi_k = \pm 1] = \frac{1}{2}$. Let M and N be two positive integers and define

$$\tau := \min\{n \geq 0 : X_n = -N \text{ or } X_n = M\}.$$

(a) Prove that τ is an \mathbb{F} -stopping time, where \mathbb{F} is the natural filtration generated by X .

(b) Assume that $\tau < +\infty$ a.s., prove that $\mathbb{P}[X_\tau \in \{-N, M\}] = 1$.

(c) Under the condition of (b), compute $\mathbb{E}[X_\tau]$ and $\mathbb{P}[X_\tau = -N]$.

Hint: Let X be a martingale and τ be a stopping time with respect to a filtration \mathbb{F} , and if $\tau < \infty$ and the process $(X_{\tau \wedge n})_{n \geq 0}$ is uniformly bounded. Then $\mathbb{E}[X_\tau] = \mathbb{E}[X_0]$.